Second and higher Order Differential Equations

Section 3.1 Second-Order Linear Equations

Def^n: A differential equation involving derivatives of highest order two. i.e. differential eq^n involving terms like $\frac{d^2y}{dx^2}$, y'' but not y''', $\frac{d^4y}{dx^4}$ etc.

Note: Equations of type $G(x, y, y', y'')=0$ are called second-order linear equations provided G is linear in the dependent variable y and its derivatives y' and y''.

Alternatively, it can also be written in the form.

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)\,y = F(x), \quad a_2(x)\neq 0 \quad \text{—(1)}$$

Eg: (1) $x^2y'' + 2xy' + 3y = \cos x$

(2) $\sin x\, y'' + 9xy' + 3xy = 0$

are examples of linear second-order differential eq^n.

Further, eq^{n}s like

(1) $y'' + 3(y')^2 + 4y^3 = 0$

(2) $y'' = yy'$

are examples of non-linear second order differential eq^n. because products and powers of y or its derivatives appear.

Homogeneous Linear equations

If $F(x)=0$ in equations of type (1), then they are known as homogeneous linear equations.

i.e. $a_2(x)y'' + a_1(x)\,y' + a_0(x)\,y = 0, \quad$ — (2)
$a_2(x)\neq 0$

eg. (1) $x^2y'' + 2xy' + 3y = 0$.

(2) $3x^2y'' + 7y' + 2y = 0$

Non-homogeneous linear Equations

Equations that are not homogeneous linear are called non homogeneous linear equations.

eg. $x^2y'' + 2xy' + 3y = \cos x$.

Dividing eq^n (2) by $a_2(x) \neq 0$, we get —

$$y'' + \frac{a_1(x)}{a_2(x)} y' + \frac{a_0(x)}{a_2(x)} y = 0$$

or, $y'' + p(x) y' + q(x) y = 0$

is another way of representing second order homogeneous linear differential eq^n where the coefficient of y'' is 1.

Similarly dividing eq^n (1) by $a_2(x) \neq 0$ we get $y'' + p(x)y' + q(x)y = f(x)$, second order non-homogeneous linear differential eq^n.

Solution of 2^nd Order (homogeneous) linear eq^n

Let $y_1(x)$ is a sol^n of 2^nd order DE of type

$$y'' + p(x)y' + q(x) y = 0 \quad \text{— (3)}$$

Then $y_1'' + p(x) y_1' + q(x) y_1 = 0$.

Again let $y_2(x)$ is also a sol^n of above DE.

Then their linear combination i.e $y = c_1y_1 + c_2y_2$ where c_1 and c_2 are constants, is also a sol^n of D.E (3). So we have the below theorem.

Theo1 Principle of Superposition for Homogeneous Equations —

Let y_1 and y_2 be two solutions of the homogeneous linear equation $y'' + p(x)y' + q(x) y = 0$. If c_1 and c_2 are constants, then the linear combination

$$y = c_1y_1 + c_2y_2$$

is also a solution of the homogeneous linear equation.

Ex 1 :- Suppose $y_1(x) = \cos x$ and $y_2(x) = \sin x$ are two solutions of the equation $y'' + y = 0$. Then their linear combination such as
$y = 3y_1(x) - 2y_2(x) = 3\cos x - 2\sin x$ is also a solution. So, we can say that every solution of $y'' + y = 0$ is a linear combination of these two particular solutions y_1 and y_2. Thus, a general solution of $y'' + y = 0$ is given by
$y(x) = c_1 \cos x + c_2 \sin x$.

Theo 2. Existence and uniqueness for Linear Equations

Suppose that the functions p, q, and f are continuous on the open interval I containing the point a. Then, given any two numbers b_0 and b_1, the equation

$$y'' + p(x)y' + q(x)y = f(x). \quad \text{—(4)}$$

has a unique solution on the entire interval I that satisfies the initial conditions $y(a) = b_0$, $y'(a) = b_1$. —(5)

Remark: Eq^n (4) together with the conditions (5) constitute a second order initial value problem. Theorem (2) tells us that any such initial value problem has a unique solution on the whole interval I where the coefficient functions in (4) are continuous.

Q1 Show that $y = 3e^{2x} + e^{-2x} - 3x$ is the unique sol^n of the Initial value problem $y'' - 4y = 12x$ where $y(0) = 4$, $y'(0) = 1$.

Sol^n. Given $y = 3e^{2x} + e^{-2x} - 3x$ —(a) and D.E. $y'' - 4y = 12x$ —(b)

Then, $y' = 6e^{2x} - 2e^{-2x} - 3$ —(c)

$y'' = 12e^{2x} + 4e^{-2x}$ —(d)

Also, $y(0) = 3 + 1 - 0 = 4$

$y'(0) = 6 - 2 - 3 = 1$.

Substituting (a), (c) & (d) in LHS of (b), we get

$12e^{2x} + 4e^{-2x} - 4(3e^{2x} + e^{-2x} - 3x)$

$= \cancel{12e^{2x}} + \cancel{4e^{-2x}} - \cancel{12e^{2x}} - \cancel{4e^{-2x}} + 12x = 12x =$ RHS of (b)

Thus, $y = 3e^{2x} + e^{-2x} - 3x$ is sol^n of D.E. $y'' - 4y = 12x$ with $y(0) = 4$ and $y'(0) = 1$.

For uniqueness :- Coefficient of y'' is 1. On comparing given DE (b) with standard form we gets

$p(x) = 0$, $q(x) = -4$, $f(x) = 12x$

Since $p(x)$, $q(x)$ and $f(x)$ are cts functions on $\mathbb{R}$. Thus using Existence and Uniqueness theorem the solution $y = 3e^{2x} + e^{-2x} - 3x$ is the unique sol^n.

Q2 :- Show that the function $y = cx^2 + x + 3$ is a solution, though not unique, of the initial value problems
$x^2y'' - 2xy' + 2y = 6$ —— (A)
with $y(0) = 3$, $y'(0) = 1$ on $(-\infty, \infty)$

Sol. Given $y = cx^2 + x + 3$ —(1) and
DE $x^2y'' - 2xy' + 2y = 6$ —(A)

we can easily verify that y given by (1) is a solⁿ of eqⁿ (A)
with $y(0) = 3$, $y'(0) = 1$

Coefficient of y'' i.e. $x^2 = 0$ at $x = 0 \in (-\infty, \infty)$. So we cannot divide eqⁿ (A) by x^2. Hence the property of general second-order differential eqⁿ cannot be attained.
Hence according to Existence and uniqueness theorem the solution is not unique.
Also, due to presence of c w.r.t. y which can take different values there are several solutions of the given differential equation.

Q3. verify that the function $y_1(x) = e^x$ and $y_2(x) = xe^x$ are solutions of the D.E
$$y'' - 2y' + y = 0 \text{ —— (A)}$$
and then find a solution satisfying the initial conditions $y(0) = 3$ and $y'(0) = 1$.

Sol: verify yourself that y_1 and y_2 are solⁿˢ of eqⁿ (A)
Since the given differential eqⁿ is homogeneous so, by the principle of superposition for homogeneous eqⁿˢ. If y_1 and y_2 are solⁿ's of DE (A) then their linear combination i.e. $y = c_1y_1 + c_2y_2$ (c_1, c_2 are constants) will also be a solution of the DE (A).

$\because$ $y(0)=3$ and $y'(0)=1$

Also, $y = c_1 y_1 + c_2 y_2 = c_1 e^x + c_2 x e^x$

$\therefore y' = c_1 e^x + c_2(e^x + x e^x)$

$= (c_1+c_2)e^x + c_2 x e^x.$

$y(0)=3 \Rightarrow c_1 e^0 + c_2 . 0 e^0 = 3 \Rightarrow c_1 = 3.$

$y'(0)=1 \Rightarrow (c_1+c_2)e^0 + c_2.0.e^0 = 1 \Rightarrow c_1+c_2=1$

$\Rightarrow c_2 = 1-c_1 = 1-3 = -2.$

Thus, $y(x) = 3e^x - 2xe^x$. is the required solution of the given DE (A).

Def^n Linearly Dependence of two functions

Two functions $u_1(x)$ and $u_2(x)$ are said to be linearly dependent if $\exists$ c_1 and c_2, not both zero such that $c_1 u_1(x) + c_2 u_2(x) = 0$.

Say $c_1 \neq 0$, then, $c_1 u_1(x) = -c_2 u_2(x)$

$\Rightarrow u_1(x) = \frac{-c_2}{c_1} u_2(x)$

$u_1(x) = \alpha u_2(x)$ where $\alpha = -\frac{c_2}{c_1}$.

Thus, ~~if~~ two functions are linearly dependent ~~if~~ If they are ~~one of them is a constant~~ multiple of ~~the other~~. each other, i.e. $\frac{u_1}{u_2}$ or $\frac{u_2}{u_1}$ is a constant valued function.

Def^n. Linearly Independence of two functions

Two functions $u_1(x)$ and $u_2(x)$ are said to be L.I. iff $c_1 u_1(x) + c_2 u_2(x) = 0 \Rightarrow c_1 = c_2 = 0$.

In other words, the two functions are LI if neither is a constant multiple of the other.

Ex. The following pair of functions are LI on the entire real line.

$\sin x$ and $\cos x$, e^x and e^{-2x}, e^x and xe^x, $x+1$ and x^2,

x and $|x|$.

Since $\frac{\sin x}{\cos x} = \tan x$ or $\frac{\cos x}{\sin x} = \cot x$ which is not a constant.

Similarly $\frac{e^x}{e^{-2x}} = e^{3x}$.

But the functions like $\sin 2x$ and $\sin x \cos x$ are LD.

Since. $\frac{\sin 2x}{\sin x \cos x} = 2$ which is constant.

Wronskian:-

The wronskian of two functions f and g is defined as —

$$W = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g.$$

Note :- We will establish a relation between wronskian and linear dependence of two functions so that we can tell whether the solution of second order differential equations are linearly dependent or independent.

Note :- The homogeneous equation $y'' + py' + qy = 0$ (A) always have two linearly independent solution say $y_1(x)$ and $y_2(x)$. If y is any solution of above DE (A). then it can always be represented as linear combinations of y_1 and y_2. This linear combination of independent solutions is known as General sol^n of Homogeneous DE of 2nd order.

Theo 3. Wronskian of Solutions

Suppose that y_1 and y_2 are two solutions of the homogeneous second-order linear differential equation

$$y'' + p(x)y' + q(x)y = 0$$

on an open interval I on which p and q are continuous.

(a) If y_1 and y_2 are linearly dependent then $W(y_1, y_2) \equiv 0$ on I.

(b) If y_1 and y_2 are linearly independent, then $W(y_1, y_2) \neq 0$ at each point of I.

Question :- If $y_1(x) = \sin 3x$ and $y_2(x) = \cos 3x$ are two solutions of $y'' + 9y = 0$, show that $y_1(x)$ and $y_2(x)$ are linearly independent sol^{n}s!

Sol^n We have $W(y_1(x), y_2(x)) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin 3x & \cos 3x \\ 3\cos 3x & -3\sin 3x \end{vmatrix}$

$$= -3\sin^2 3x - 3\cos^2 3x = -3(\sin^2 3x + \cos^2 3x) = -3. \neq 0.$$

Hence y_1 and y_2 are L.I sol^{n}s of $y'' + 9y = 0$.

Question : Show that linearly independent sol^{n}s of $y'' - 2y' + 2y = 0$ are $e^x \sin x$ and $e^x \cos x$. What is the general sol^n?. Find the sol^n $y(x)$ with the property $y(0) = 2$ and $y'(0) = -3$.

Sol^n, Let $y_1 = e^x \sin x$ and $y_2 = e^x \cos x$

Then $y_1' = e^x \sin x + e^x \cos x$, $y_2' = e^x \cos x - e^x \sin x$.

$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x \sin x & e^x \cos x \\ e^x \sin x + e^x \cos x & e^x \cos x - e^x \sin x \end{vmatrix}$

$$= \cancel{e^{2x} \sin x \cos x} - e^{2x} \sin^2 x - \cancel{e^{2x} \sin x \cos x} - e^{2x} \cos^2 x = -e^{2x} \neq 0.$$

$\therefore$ y_1 and y_2 are linearly independent.

Let $y = c_1 y_1 + c_2 y_2 = c_1 e^x \sin x + c_2 e^x \cos x$ is the general sol^n of given DE (where c_1 and c_2 are arbitrary constants).

Then

$y(0) = 2 \Rightarrow c_1 e^0 \sin 0 + c_2 e^0 \cos 0 = 2$

$\Rightarrow 0 + c_2 = 2 \Rightarrow c_2 = 2.$

$y'(0) = -3 \therefore c_1 e^x \sin x + c_1 e^x \cos x + c_2 e^x \cos x - c_2 e^x \sin x = y'(x)$

$\therefore y'(0) = c_1 + c_2 = -3. \Rightarrow c_1 = -3 - c_2 = -3 - 2 = -5$

$\therefore y = -5 e^x \sin x + 2 e^x \cos x$ is the required general solution.

Theorem 4 :- General Solutions of Homogeneous Equations

let y_1 and y_2 be two linearly independent solutions of the homogeneous equation $y'' + p(x) y'(x) + q(x) y = 0$ ——(A) with p and q ~~cts~~ on the open interval I. If Y is any solution whatsoever of eq^n (A) on I, then $\exists$ nos c_1 and c_2 such that $Y(x) = c_1 y_1(x) + c_2 y_2(x)$ $\forall x \in I$.

Proof :- Choose a point a on I and consider the simultaneous equations.

$$c_1 y_1(a) + c_2 y_2(a) = Y(a) \qquad \text{———— (6)}$$
$$c_1 y_1'(a) + c_2 y_2'(a) = Y'(a)$$

The determinant of the coefficients in this system of linear equations in the unknowns c_1 and c_2 is the Wronskian $W(y_1, y_2)$ evaluated at a. By theorem 3. this determinant is nonzero, so the eq^{n}s in (6) can be solved for c_1 and c_2. So, with these values of c_1 and c_2 we define the solution $G(x) = c_1 y_1(x) + c_2 y_2(x)$ of eq^n (A) then

$$G(a) = c_1 y_1(a) + c_2 y_2(a) = Y(a)$$
$$G'(a) = c_1 y_1'(a) + c_2 y_2'(a) = Y'(a).$$

Thus the two solutions Y and G have the same initial values at a as so Y' and G'. By the uniqueness of solution determined by initial values (Theorem 2). It follows that Y and G agree on I.

Hence $Y(x) = G(x) = c_1 y_1(x) + c_2 y_2(x)$.

Example. If $y_1 = e^{2x}$ and $y_2 = e^{-2x}$, Then

$$y_1'' = (2)(2)e^{2x} = 4e^{2x} = 4y_1 \quad \text{and} \quad y_2'' = (-2)(-2)e^{-2x} = 4e^{-2x} = 4y_2.$$

$\therefore$ y_1 and y_2 are L.I solutions of $y'' - 4y = 0$ ——— (i)

But $y_3(x) = \cosh 2x$ and $y_4(x) = \sinh 2x$ are also solⁿ of eqⁿ (1)

because $\frac{d^2}{dx^2}(\cosh 2x) = 4\cosh 2x = 4y_3$

and similarly $\frac{d^2}{dx^2}(\sinh 2x) = 4\sinh 2x = 4y_4$. Thus, by theorem 4

$\cosh 2x$ and $\sinh 2x$ can be expressed as linear combinations of

$y_1(x) = e^{2x}$ and $y_2(x) = e^{-2x}$.

i.e. $\cosh 2x = c_1 e^{2x} + c_2 e^{-2x} = \frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x}$ [where $c_1 = c_2 = \frac{1}{2}$]

$\sinh 2x = c_3 e^{2x} + c_4 e^{-2x} = \frac{1}{2}e^{2x} - \frac{1}{2}e^{-2x}$. [where $c_1 = \frac{1}{2}, c_2 = -\frac{1}{2}$]